A SOLUTION TO THE FOUR-DOGS PURSUIT PROBLEM BY A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS AND A DESCRIPTION OF INSTANTANEOUS POSITION DURING THE PROCESS BASED ON THE SOLUTION

Abstract. In this article, we give a new solution of the Four-Dogs Pursuit Problem by a system of linear ordinary differential equations with constant coefficients. First, we set up the system of linear ordinary differential equations with constant coefficients based on the fact that the trajectories of the four dogs have nothing to do with their speed if the speed of them are equal to each other at arbitrary instantaneous time. Then, we worked out the trajectory of each dog by the eigenvalues and eigenvectors of the coefficient matrix of the system of linear ordinary differential equations. Then, we calculated the length of the trajectory along with the time needed for the whole process and described the instantaneous position of each dog during the pursuing process. Finally, the MATLAB Code of plotting the trajectories of the four dogs is given in the Appendix.

The Four-Dogs Pursuit Problem is demonstrated as follows: Four dogs at the vertices A, B, C and D of a square with side length L (Figure 1) begin running directly after the dog on his right side at the same speed v. Without loss of generality, We assume that the initial positions of Dogs are $A(\frac{L}{2}, \frac{L}{2})$, $B(-\frac{L}{2}, \frac{L}{2})$, $C(-\frac{L}{2}, -\frac{L}{2})$, $D(\frac{L}{2}, -\frac{L}{2})$ in O-xy coordinates and A pursues B, B pursues C, C pursues D, D pursues A. Since the directions of each dog's velocity are always straight forward to the dog being pursued, the dogs will spiral around each other as their directions keep changing and by symmetry, we know that they will end up meeting at the original point $(0, 0)$, the middle of the square. What is the path length walked by each dog?

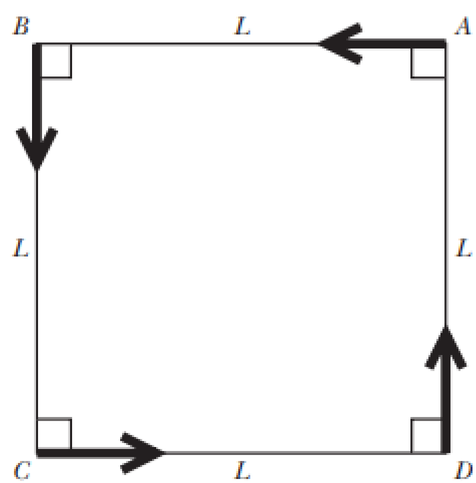

Figure 1. The square dog problem.

There happened to be a sketch of the method being described in this article for a similar problem on https://math.stackexchange.com/q/1889438. However, I'm going to describe the detailed solution in my own words, rather than enriching that sketch because my solution is worked out independently, without reading that sketch.

It can be claimed that the trajectories of the four dogs have nothing to do with their speed since the speed of them are equal to each other at arbitrary instantaneous time. Thus, we can introduce a parameter t which can be regarded as "adjusted time" varying from 0 to $+\infty$ and assume

$$A(x_1(t), y_1(t)),\ B(x_2(t), y_2(t)),\ C(x_3(t), y_3(t)),\ D(x_4(t), y_4(t))$$

are the coordinates of instantaneous positions of the four dogs during the whole process of their pursuit respectively, where $x_j : [0, +\infty) \mapsto \mathbb{R}$ and $y_j : [0, +\infty) \mapsto \mathbb{R}$ are continuous mappings from $[0, +\infty)$ to $\mathbb{R}$ for each $j \in \{1, 2, 3, 4\}$.

So, it can be assumed by symmetry that

$$\begin{aligned} v_A(t) &= K(t)\sqrt{(x_2(t) - x_1(t))^2 + (y_2(t) - y_1(t))^2} \\ v_B(t) &= K(t)\sqrt{(x_3(t) - x_2(t))^2 + (y_3(t) - y_2(t))^2} \\ v_C(t) &= K(t)\sqrt{(x_4(t) - x_3(t))^2 + (y_4(t) - y_3(t))^2} \\ v_D(t) &= K(t)\sqrt{(x_1(t) - x_4(t))^2 + (y_1(t) - y_4(t))^2} \end{aligned}$$

where $v_A(t)$, $v_B(t)$, $v_C(t)$, $v_D(t)$ is the speed of Dog A, B, C and D respectively and $K(t)$ is an adapted scalar function of t that makes $v_A(t)$, $v_B(t)$, $v_C(t)$ and $v_D(t)$ keep v.

And since the trajectories have nothing to do with speed, the adapted scalar function $K(t)$ can just be regarded as a positive real number k. Therefore, it can be figured out that

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix} = k \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ x_3 - x_2 \\ y_3 - y_2 \\ x_4 - x_3 \\ y_4 - y_3 \\ x_1 - x_4 \\ y_1 - y_4 \end{bmatrix} = k \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix} \triangleq \mathbf{P} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix} \quad (*)$$

And

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix} (0) = \begin{bmatrix} \frac{L}{2} \\ \frac{L}{2} \\ -\frac{L}{2} \\ \frac{L}{2} \\ -\frac{L}{2} \\ -\frac{L}{2} \\ \frac{L}{2} \\ -\frac{L}{2} \end{bmatrix} \quad (\star)$$

For the eigenpolynomial of the coefficient matrix $\mathbf{P}$ of System $(*)$, we have

$$|\mathbf{P} - \lambda\mathbf{E}| = [(\lambda + k)^4 - k^4]^2 = \lambda^2(\lambda + 2k)^2(\lambda^2 + 2k\lambda + 2k^2)^2$$

So it can be worked out that the eigenvalues of $\mathbf{P}$ are $0, -2k, -k \pm ki$ and the eigenvectors respect to them are

$$\boldsymbol{\xi}_{0,1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{\xi}_{0,2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{\xi}_{-2k,1} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{\xi}_{-2k,2} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix},$$

$$\boldsymbol{\xi}_{-k+ki,1} = \begin{bmatrix} i \\ 0 \\ -1 \\ 0 \\ -i \\ 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{\xi}_{-k+ki,2} = \begin{bmatrix} 0 \\ i \\ 0 \\ -1 \\ 0 \\ -i \\ 0 \\ 1 \end{bmatrix} \text{ and } \boldsymbol{\xi}_{-k-ki,1} = \bar{\boldsymbol{\xi}}_{-k+ki,1}, \boldsymbol{\xi}_{-k-ki,2} = \bar{\boldsymbol{\xi}}_{-k+ki,2}$$

Denote $\boldsymbol{\eta}_1 = \boldsymbol{\xi}_{0,1}, \boldsymbol{\eta}_2 = \boldsymbol{\xi}_{0,2}, \boldsymbol{\eta}_3 = e^{-2t}\boldsymbol{\xi}_{-2k,1}, \boldsymbol{\eta}_4 = e^{-2t}\boldsymbol{\xi}_{-2k,2}$, and

$$\boldsymbol{\eta}_5 = e^{-kt} \begin{bmatrix} -\sin kt \\ 0 \\ -\cos kt \\ 0 \\ \sin kt \\ 0 \\ \cos kt \\ 0 \end{bmatrix}, \boldsymbol{\eta}_6 = e^{-kt} \begin{bmatrix} \cos kt \\ 0 \\ -\sin kt \\ 0 \\ -\cos kt \\ 0 \\ \sin kt \\ 0 \end{bmatrix},$$

$$\boldsymbol{\eta}_7 = e^{-kt} \begin{bmatrix} 0 \\ -\sin kt \\ 0 \\ -\cos kt \\ 0 \\ \sin kt \\ 0 \\ \cos kt \end{bmatrix}, \boldsymbol{\eta}_8 = e^{-kt} \begin{bmatrix} 0 \\ \cos kt \\ 0 \\ -\sin kt \\ 0 \\ -\cos kt \\ 0 \\ \sin kt \end{bmatrix}.$$

Then we know that the general solution of System $(*)$ is

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix} = c_1\boldsymbol{\eta}_1 + c_2\boldsymbol{\eta}_2 + c_3\boldsymbol{\eta}_3 + c_4\boldsymbol{\eta}_4 + c_5\boldsymbol{\eta}_5 + c_6\boldsymbol{\eta}_6 + c_7\boldsymbol{\eta}_7 + c_8\boldsymbol{\eta}_8.$$

According to the initial values provided by $(\star)$, it can be worked out that $c_1 = c_2 = c_3 = c_4 = 0, c_5 = c_6 = c_8 = \frac{L}{2}, c_7 = -\frac{L}{2}$. So the trajectories of the four dogs A, B, C and D can be described by the following parametric equations as

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix}(t) = \frac{Le^{-kt}}{2} \begin{bmatrix} \cos kt - \sin kt \\ \cos kt + \sin kt \\ -\cos kt - \sin kt \\ \cos kt - \sin kt \\ -\cos kt + \sin kt \\ -\cos kt - \sin kt \\ \cos kt + \sin kt \\ -\cos kt + \sin kt \end{bmatrix}$$

The following Figure 2 shows a plot of this process with the example of $L = 2$ and $k = 1$ and the MATLAB Code for Figure 2 is shown in the attachment.

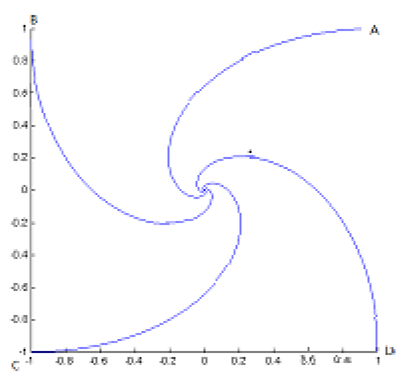

Figure 2. Trajectories of Dogs

Since $\int_0^{+\infty} \sqrt{(\frac{dx_j}{dt})^2 + (\frac{dy_j}{dt})^2} dt = L$ for each $j \in \{1, 2, 3, 4\}$, the distance travelled by each dog is L and the time needed is $\frac{L}{v}$.

Now let's turn to the description of instantaneous positions during the process of the pursuit. It is obvious that for arbitrary instantaneous time $\tau \in (0, \frac{L}{v})$, the distance passed by each dog is $v\tau$ and that $v\tau \in (0, L)$. Thus, if we define a function

$$s_j : [0, +\infty) \mapsto \mathbb{R}$$

for each $j \in \{1, 2, 3, 4\}$ such that

$$s_j(z) = \int_0^z \sqrt{(\frac{dx_j}{dt})^2 + (\frac{dy_j}{dt})^2} dt = L(1 - e^{-kz})$$

for arbitrary $z \in [0, +\infty)$, then $\lim_{z \to +\infty} s_j(z) = L$ for each $j \in \{1, 2, 3, 4\}$. By Intermediate Value Theorem and the Precise Definition of Limits, we know that there exists some $\xi = \frac{1}{k} \ln \frac{L}{L - v\tau} > 0$ such that $s_j(\xi) = v\tau$. Therefore, at the instantaneous time τ, the positions of the four dogs are $A_\tau(x_1(\xi), y_1(\xi))$, $B_\tau(x_2(\xi), y_2(\xi))$, $C_\tau(x_3(\xi), y_3(\xi))$ and $D_\tau(x_4(\xi), y_4(\xi))$ respectively, i.e.

$$A(\tfrac{L-v\tau}{2}(\cos\ln\tfrac{L}{L-v\tau} - \sin\ln\tfrac{L}{L-v\tau}), \tfrac{L-v\tau}{2}(\cos\ln\tfrac{L}{L-v\tau} + \sin\ln\tfrac{L}{L-v\tau}))$$
$$B(\tfrac{L-v\tau}{2}(-\cos\ln\tfrac{L}{L-v\tau} - \sin\ln\tfrac{L}{L-v\tau}), \tfrac{L-v\tau}{2}(\cos\ln\tfrac{L}{L-v\tau} - \sin\ln\tfrac{L}{L-v\tau}))$$
$$C(\tfrac{L-v\tau}{2}(-\cos\ln\tfrac{L}{L-v\tau} + \sin\ln\tfrac{L}{L-v\tau}), \tfrac{L-v\tau}{2}(-\cos\ln\tfrac{L}{L-v\tau} - \sin\ln\tfrac{L}{L-v\tau}))$$
$$D(\tfrac{L-v\tau}{2}(\cos\ln\tfrac{L}{L-v\tau} + \sin\ln\tfrac{L}{L-v\tau}), \tfrac{L-v\tau}{2}(-\cos\ln\tfrac{L}{L-v\tau} + \sin\ln\tfrac{L}{L-v\tau}))$$

Appendix

MATLAB Code for Figure 2

```
M=5999;
s=0.001;
n=M/s+1;
t=linspace(0,M,n);
x1=linspace(0,M,n);
y1=linspace(0,M,n);
x2=linspace(0,M,n);
y2=linspace(0,M,n);
x3=linspace(0,M,n);
y3=linspace(0,M,n);
x4=linspace(0,M,n);
y4=linspace(0,M,n);
i=1;
while i<=n
   x1(i)=exp(-t(i))*(cos(t(i))-sin(t(i)));
   y1(i)=exp(-t(i))*(cos(t(i))+sin(t(i)));
   x2(i)=exp(-t(i))*(-cos(t(i))-sin(t(i)));
   y2(i)=exp(-t(i))*(cos(t(i))-sin(t(i)));
   x3(i)=exp(-t(i))*(-cos(t(i))+sin(t(i)));
   y3(i)=exp(-t(i))*(-cos(t(i))-sin(t(i)));
   x4(i)=exp(-t(i))*(cos(t(i))+sin(t(i)));
   y4(i)=exp(-t(i))*(-cos(t(i))+sin(t(i)));
   i=i+1;
end
hold;
plot(x1,y1);
plot(x2,y2);
plot(x3,y3);
plot(x4,y4);
```