Vertex-critical graphs in co-gem-free graphs

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(Joint work with Iain Beaton)

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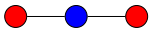
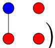
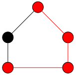
Definitions

- P_n is the path on n vertices (P_3: ).
- $G + H$ denotes the disjoint union of graphs G and H.
- $\ell G = \underbrace{G + G + \cdots + G}_{\ell}$ ($P_2 + 2P_1$: ).
- Colouring here means proper colouring (adjacent vertices get different colours).
- A graph is H-free if it does not contain H as an **induced subgraph**.  is P_5-free but not P_4-free.

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- So assuming P≠NP, if k-Colouring can be solved in polynomial-time for H-free graphs, then every component of H must be a path.
- It remains NP-complete when restricted to P_6-free graphs for all $k \geq 5$ (Huang 2016).

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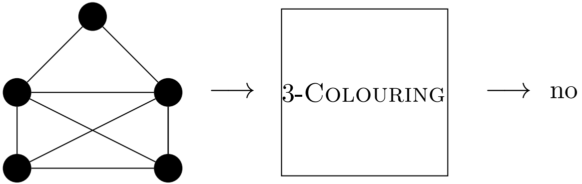

- A k-colouring is a certificate to verify a “yes”.
- How can we verify a “no”?

- A graph G is k-critical if G is not $(k-1)$-colourable, but every proper induced subgraph of G is.

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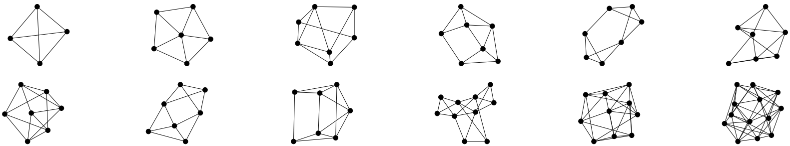
Issue: For $k \geq 3$ there are infinitely many k-critical graphs.

Theorem (Bruce-Hoàng-Sawada 2009, Maffray-Morel 2012): All twelve 4-critical P_5-free graphs are given below.



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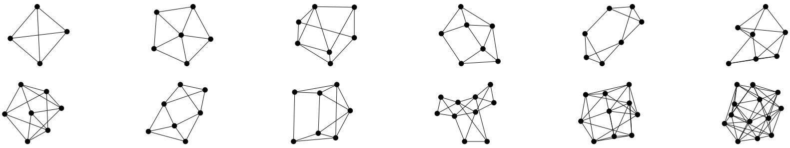

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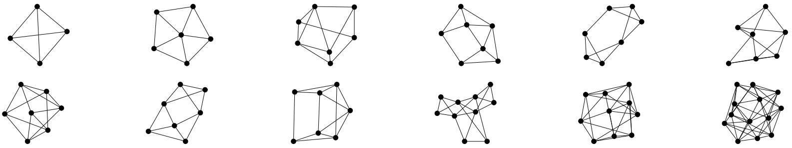

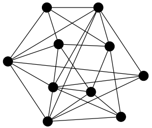

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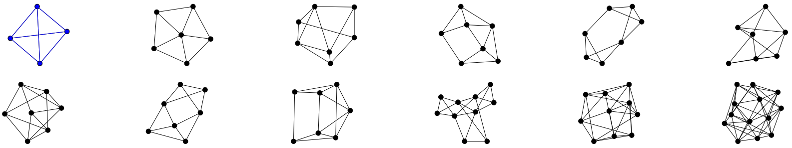

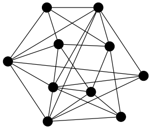

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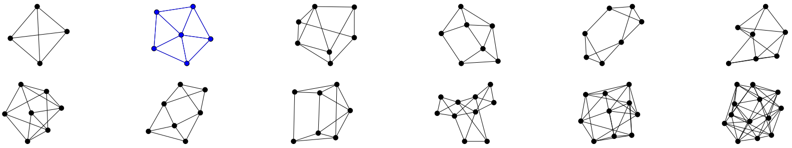

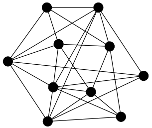

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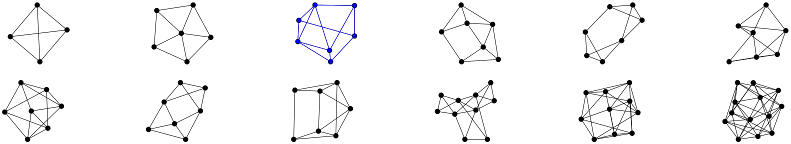

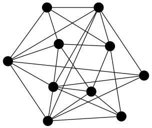

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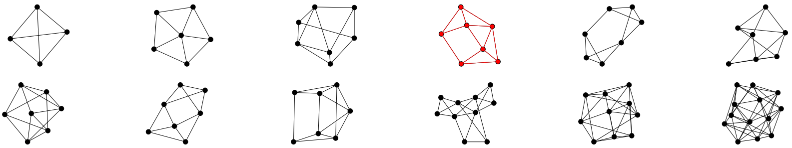

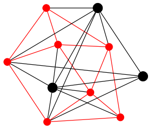

Theorem (Hoàng et al. 2015): If H contains an induced $2P_2$, then there is an infinite number of k-critical H-free graphs for all $k \geq 5$. In particular, there is an infinite number of P_t-free k-critical graphs for all $k, t \geq 5$.

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Theorem (Chudnovsky-Goedgebeur-Schaudt-Zhong 2020): There are only finitely many 4-critical H-free graphs if and only if H is an induced subgraph of P_6, $2P_3$, or $P_4 + \ell P_1$ for some $\ell \geq 0$.

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Figure: The graphs gem (left) and co-gem$=P_4 + P_1$ (right).

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Question 4: For which graphs H are there only finitely many k-critical (co-gem, H)-free graphs for all k?

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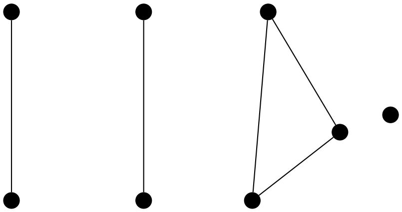
Theorem (K. Cameron-Goedgebeur-Huang-Shi 2021): For H of order 4 and $k \geq 5$, there are only finitely many k-critical (P_5, H)-free graphs if and only if H is **NOT** $2P_2$ or $K_3 + P_1$.



(a) $2P_2$ (b) $K_3 + P_1$

Open Problem (K. Cameron-Goedgebeur-Huang-Shi 2021): For which graphs H of **order 5** are there only finitely many k-critical (P_5, H)-free graphs for all k?

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- banner
- $K_{2,3}$ or $K_{1,4}$
- $P_2 + 3P_1$
- $P_3 + 2P_1$
- W_4
- $\overline{P_5}$
- $\overline{P_3 + P_2}$ or gem
- dart
- $K_{1,3} + P_1$ or $\overline{K_3 + 2P_1}$

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- $P_2 + 3P_1$ (C.-Hoàng-Sawada 2022)
- $P_3 + 2P_1$ (Abuadas-C.-Hoàng-Sawada 2024)
- W_4 (Xia-Jooken-Goedgebeur-Huang-Beaton-C. 2024+)
- $\overline{P_5}$ (Dhaliwal-Hamel-Hoàng-Maffray-McConnel-Panait 2017)
- $\overline{P_3 + P_2}$ or gem (Cai-Goedgebeur-Huang 2023)
- dart (Xia-Jooken-Goedgebeur-Huang 2023)
- $K_{1,3} + P_1$ or $\overline{K_3 + 2P_1}$ (Xia-Jooken-Goedgebeur-Huang 2024)

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Theorem (Abuadas-C.-Hoàng-Sawada 2024) There are only finitely many k-critical (gem, co-gem)-free graphs for all k.

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Main Result

Theorem (Beaton-C. 2024+) There are only finitely many k-critical (co-gem, H)-free graphs for all k and all H of order 4.

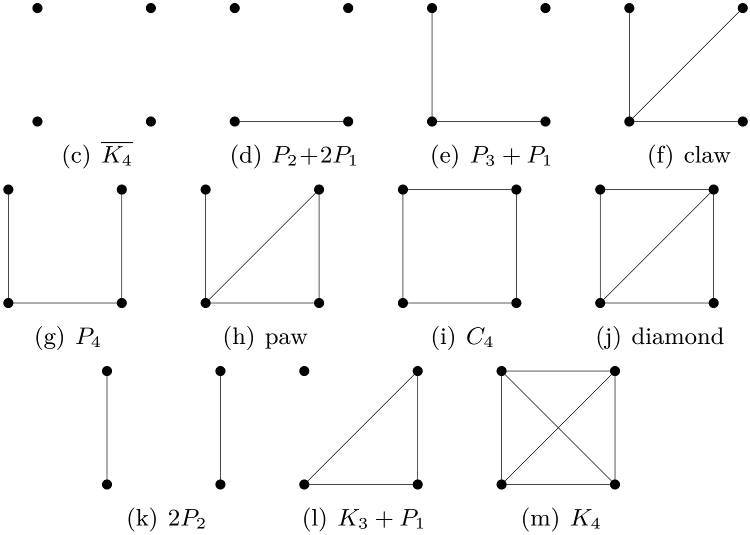
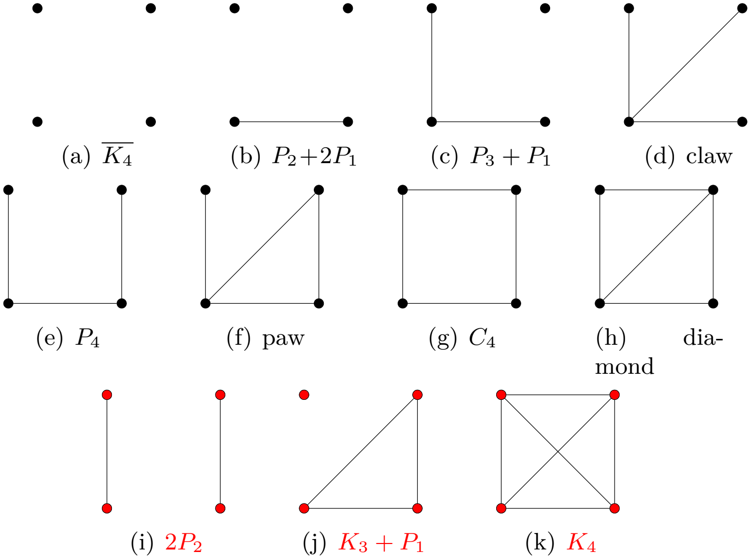


Figure: All 11 nonisomorphic graphs of order four.



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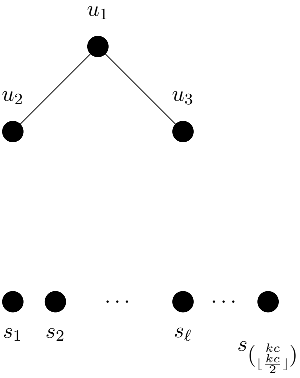

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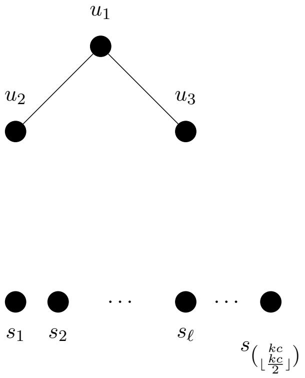
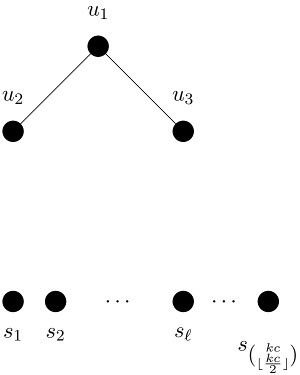


- Suppose G contains an induced $P_3 + \binom{kc}{\lfloor \frac{kc}{2} \rfloor} P1$.
- Use that G is co-gem-free to find a set U such that $N(s_i) \cap U$ is incomparable with $N(s_j) \cap U$ for all $i \neq j$.

Theorem (Beaton-C. 2024+): There are only finitely many k-critical (co-gem, $P_5, P_3 + cP_2$)-free graphs for all $k, c \geq 0$.



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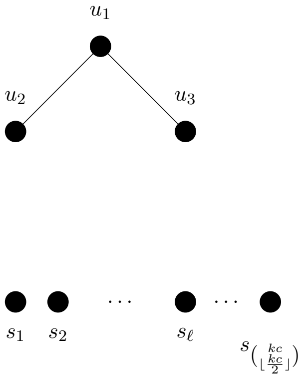
- By Sperner's Theorem on antichains, $|U| \geq kc$.

Theorem (Beaton-C. 2024+): There are only finitely many k-critical (co-gem, $P_5, P_3 + cP_2$)-free graphs for all $k, c \geq 0$.



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- Show that if $\alpha(U) > c$, then G has an induced P_5 or $P_3 + cP_2$.

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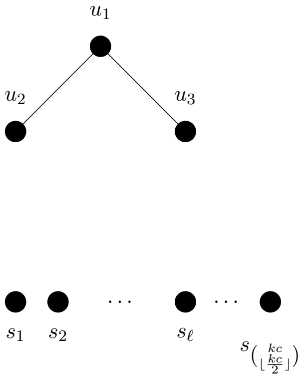

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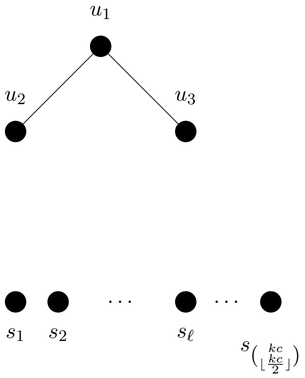

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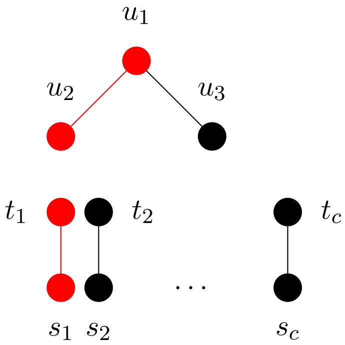
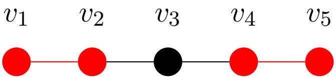
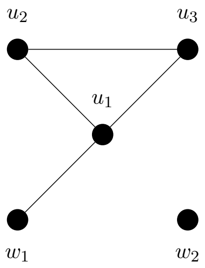
- Show that if $\alpha(U) > c$, then G has an induced P_5 or $P_3 + cP_2$.
- But now, $\chi(U) \geq k$, contradicting G being k-critical!
- So, G is $(P_3 + \binom{kc}{\lfloor \frac{kc}{2} \rfloor} P1)$-free.

Corollary (Beaton-C. 2024): There are only finitely many k-critical (co-gem, $2P_2$)-free graphs for all k.

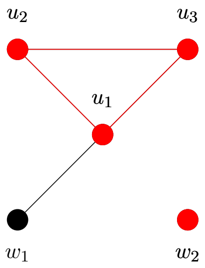




Theorem (Beaton-C. 2024+): There are only finitely many k-critical (co-gem, paw + P_1)-free graphs for all k.



Theorem (Beaton-C. 2024+): There are only finitely many k-critical (co-gem, paw $+ P_1$)-free graphs for all k.



Corollary (Beaton-C. 2024): There are only finitely many k-critical (co-gem, $K_3 + P_1$)-free graphs for all k.

Computer Proof (Beaton-C. 2024+, thanks to Jan Goedgebeur!): There are **no** 5-critical (co-gem, K_4)-free graphs.

Corollary (Beaton-C. 2024): (co-gem, K_4)-free graphs are 4-colourable.

Question 2: For which values of ℓ are there only finitely many k-critical $(P_4 + \ell P_1)$-free graphs for all k?

Question 3: Are there only finitely many k-critical co-gem-free graphs for all k?

Question 4: For which graphs H of order at least 5 are there only finitely many k-critical (co-gem, H)-free graphs for all k?

Thank You!

Figure: Scan QR code for the arXiv paper. Thanks to NSERC!