Efficient certifying algorithms for k-colouring subfamilies of P_5-free graphs

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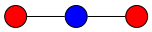
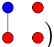
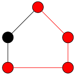
Definitions

- P_n is the path on n vertices (P_3: ).
- $G + H$ denotes the disjoint union of graphs G and H.
- $\ell G = \underbrace{G + G + \cdots + G}_{\ell}$ ($P_2 + 2P_1$: ).
- Colouring here means proper colouring (adjacent vertices get different colours).
- A graph is H-free if it does not contain H as an **induced subgraph**.  is P_5-free but not P_4-free.

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- So assuming P$\neq$NP, if k-Colouring can be solved in polynomial-time for H-free graphs, then every component of H must be a path.
- It remains NP-complete when restricted to P_6-free graphs for all $k \geq 5$ (Huang 2016).

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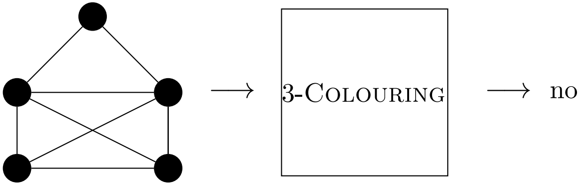

- A k-colouring is a certificate to verify a "yes".
- How can we verify a "no"?

- A graph G is k-critical if G is not $(k-1)$-colourable, but every proper induced subgraph of G is.

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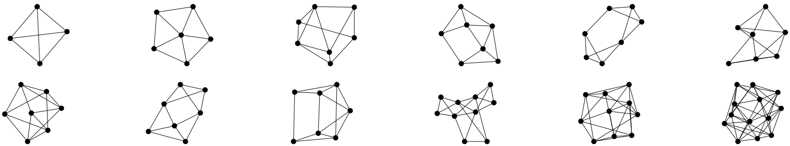
Issue: For $k \geq 3$ there are infinitely many k-critical graphs.

Theorem (Bruce-Hoàng-Sawada 2009, Maffray-Morel 2012): All twelve 4-critical P_5-free graphs are given below.



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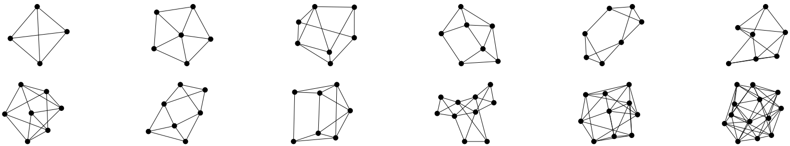

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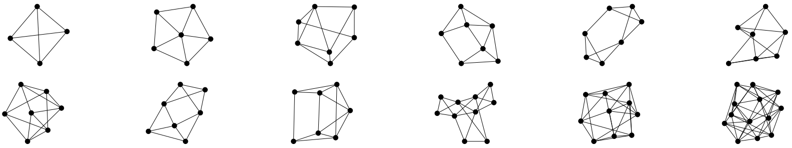

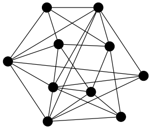

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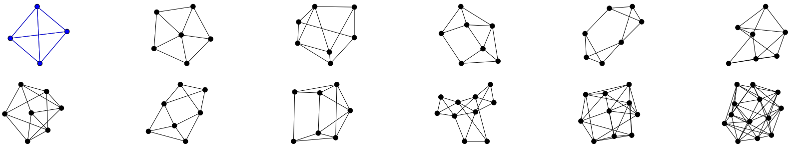

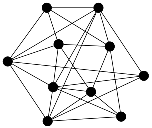

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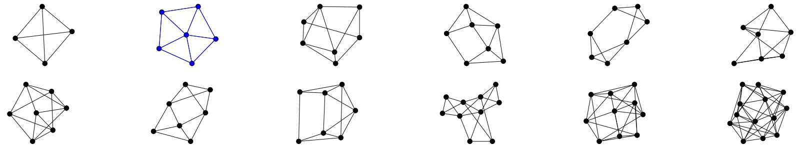

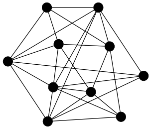

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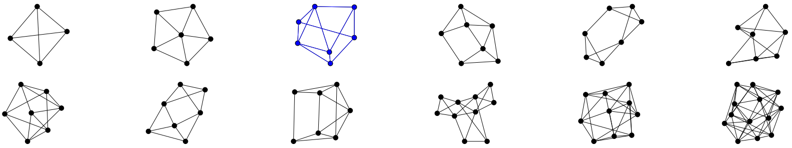

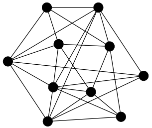

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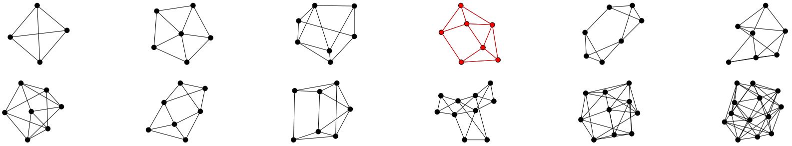

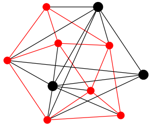

Reasonable Question: Are there only finitely many k-critical P_5-free graphs for all k?

~~Question: Are there only finitely many k-critical P_5-free graphs for all k?~~ NO!

Theorem (Hoàng-Moore-Recoskie-Sawada-Vatshelle 2015): There are infinitely many k-critical P_5-free graphs for all $k \geq 5$.

~~Question: Are there only finitely many k-critical P_5-free graphs for all k?~~ NO!

Theorem (Hoàng-Moore-Recoskie-Sawada-Vatshelle 2015): There are infinitely many k-critical P_5-free graphs for all $k \geq 5$.

Question 1: For which graphs H are there only finitely many k-critical (P_5, H)-free graphs for all k?

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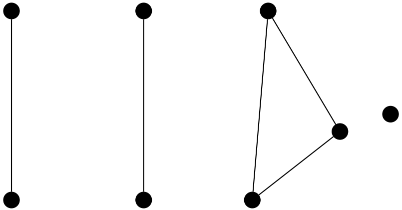
Theorem (K. Cameron-Goedgebeur-Huang-Shi 2021): For H of order 4 and $k \geq 5$, there are only finitely many k-critical (P_5, H)-free graphs if and only if H is **NOT** $2P_2$ or $K_3 + P_1$.



(a) $2P_2$ (b) $K_3 + P_1$

Open Problem (K. Cameron-Goedgebeur-Huang-Shi 2021): For which graphs H of **order 5** are there only finitely many k-critical (P_5, H)-free graphs for all k?

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Finite if H is any of the graphs below:

- banner
- $K_{2,3}$ or $K_{1,4}$
- $P_2 + 3P_1$
- $P_3 + 2P_1$
- $\overline{P_5}$
- $\overline{P_3 + P_2}$ or gem
- dart
- $K_{1,3} + P_1$ or $\overline{K_3 + 2P_1}$

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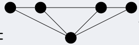
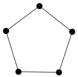
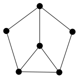
- banner (Brause-Geißer-Schiermeyer 2022)
- $K_{2,3}$ or $K_{1,4}$ (Kamiński-Pstrucha 2019)
- $P_2 + 3P_1$ (C.-Hoàng-Sawada 2022)
- $P_3 + 2P_1$ (Abuadas-C.-Hoàng-Sawada 2024)
- $\overline{P_5}$ (Dhaliwal-Hamel-Hoàng-Maffray-McConnel-Panait 2017)
- $\overline{P_3 + P_2}$ or gem (Cai-Goedgebeur-Huang 2023)
- dart (Xia-Jooken-Goedgebeur-Huang 2023)
- $K_{1,3} + P_1$ or $\overline{K_3 + 2P_1}$ (Xia-Jooken-Goedgebeur-Huang 2024)

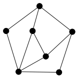
Theorem (Chudnovsky-Karthick-Maceli-Maffray 2020): If G is a $(P_5,$ gem$)$-free graph, then G is either perfect, in $\mathcal{G}_i$ for some $i \in \{1, \dots, 10\}$, or $G \in \mathcal{H}$. (gem= )



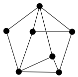
(a) G_1



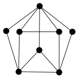
(b) G_2



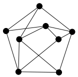
(c) G_3



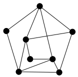
(d) G_4



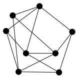
(e) G_5



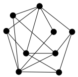
(f) G_6



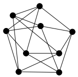
(g) G_7



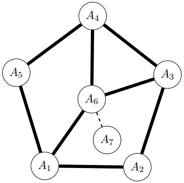
(h) G_8



(i) G_9



(j) G_{10}



(k) Graphs in $\mathcal{H}$

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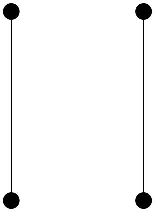
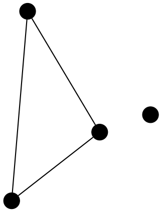
Theorem (Hoàng-Moore-Recoskie-Sawada-Vatshelle 2015): There are infinitely many k-critical (P_5, H)-free graphs for all $k \geq 5$ if $H = 2P_2$ or $K_3 + P_1$.



(a) $2P_2$



(b) $K_3 + P_1$

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Theorem (Hoàng-Moore-Recoskie-Sawada-Vatshelle 2015)**:** There are only finitely many many 5-critical (P_5, C_5)-free graphs. (In fact, exactly 13)

Theorem (C.-Hoàng 2024): For all $q \geq 1$ and $k \geq 5$ there is a $(k+1)$-critical (P_5, C_5)-free graphs of order $qk+1$ (infinitely many).

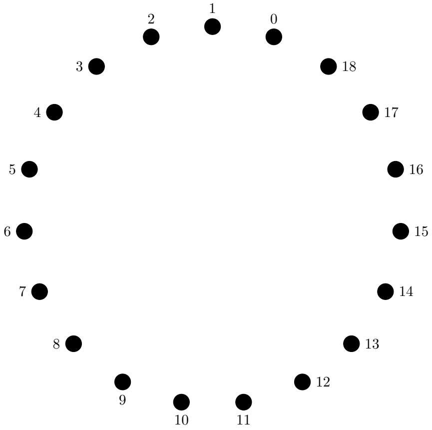

Figure: Constructing and colouring the 7-critical graph $G(3,6)$.

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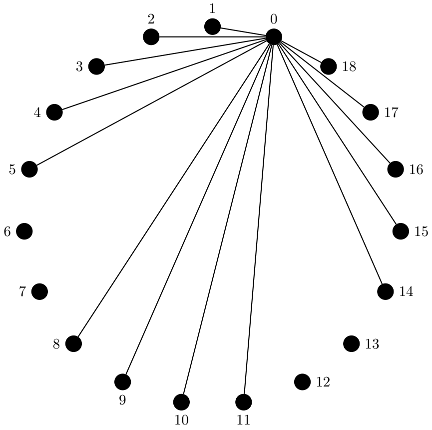

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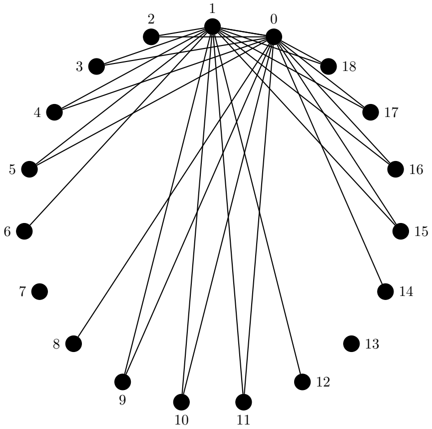

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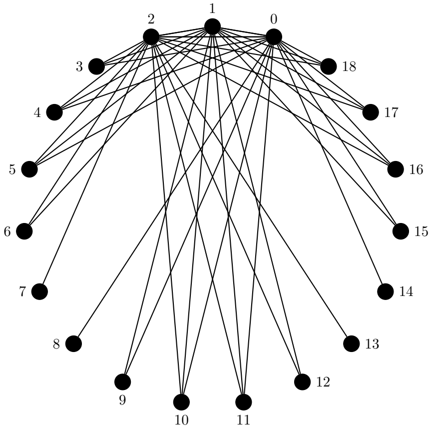

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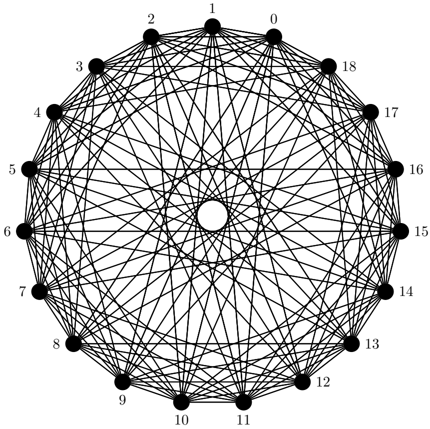

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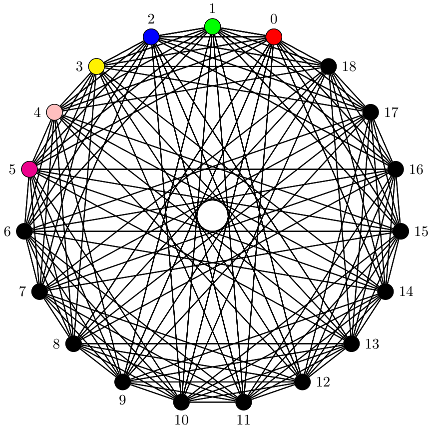

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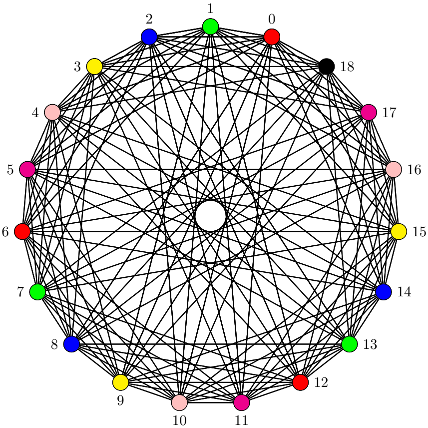

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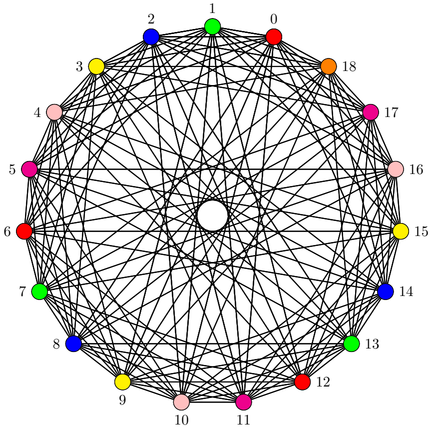

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Theorem (Hoàng-Moore-Recoskie-Sawada-Vatshelle 2015): There are infinitely many k-critical P_5-free graphs for all $k \geq 5$.

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Theorem (Hoàng-Moore-Recoskie-Sawada-Vatshelle 2015): There are infinitely many k-critical P_5-free graphs for all $k \geq 5$.

They prove this by constructing an infinite family of k-critical $(2P_2, K_3 + P_1)$-free graphs!

Figure: P_5 with an induced $2P_2$ in red.

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Fact: Every known infinite family of k-critical P_5-free graphs is actually $2P_2$-free!

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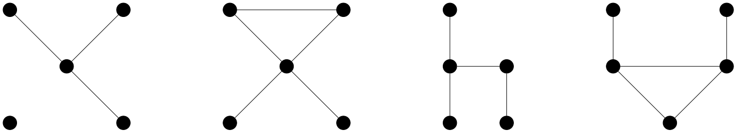
Open Problem (K. Cameron-Goedgebeur-Huang-Shi 2021): For which graphs H of **order 5** are there only finitely many k-critical (P_5, H)-free graphs for all k?



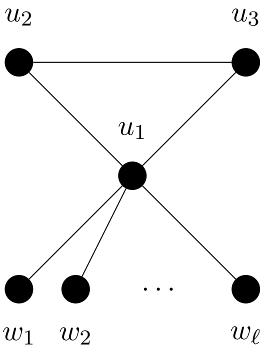
(a) *claw* $+ P_1$ (b) *cricket* (c) *chair* (d) *bull*

Figure: Graphs of order 5 where the finiteness is unknown.

(a) $m = 3$.

(b) $m = 4$.

Figure: The general form of the (m, ℓ)-*squid* graphs for $m = 3, 4$.



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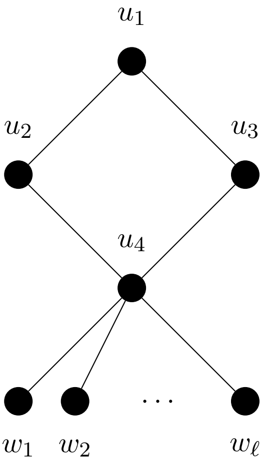

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Lemma (Adekanye-Bury-C.-Knodel 2024): Let $\ell, k \geq 1$ and $c = (\ell - 1)(k - 1) + 1$. If G is a k-critical ($2P_2$, $(4, \ell)$-*squid*)-free graph, then G is $(P_3 + cP_1)$-free.

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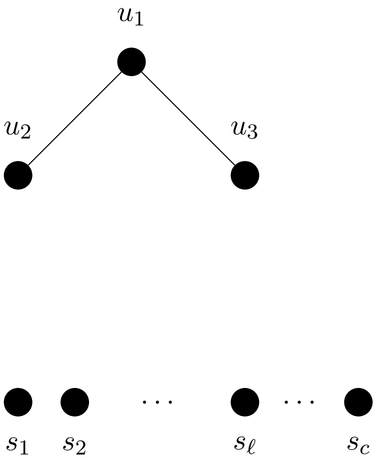

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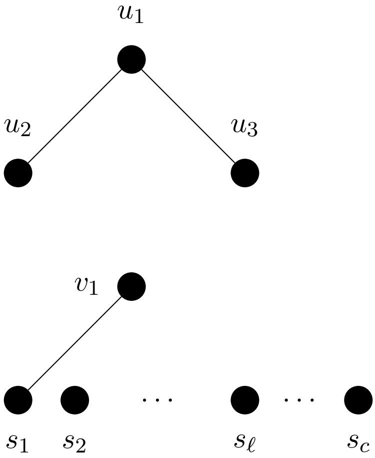

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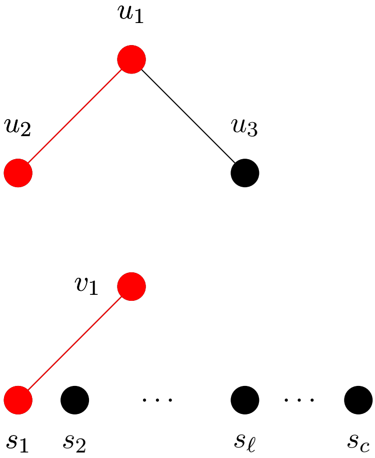

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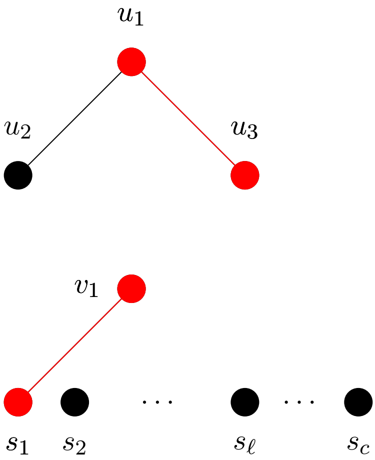

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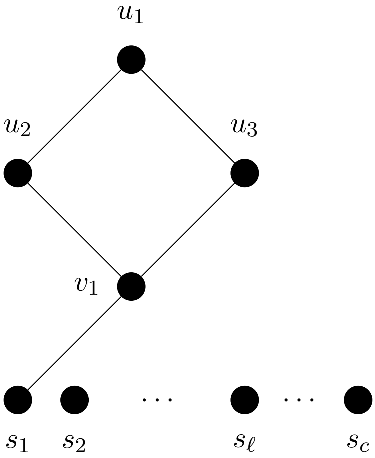

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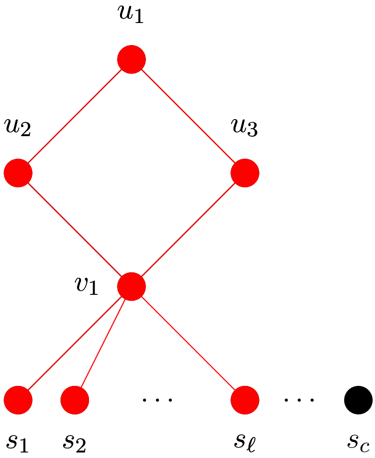

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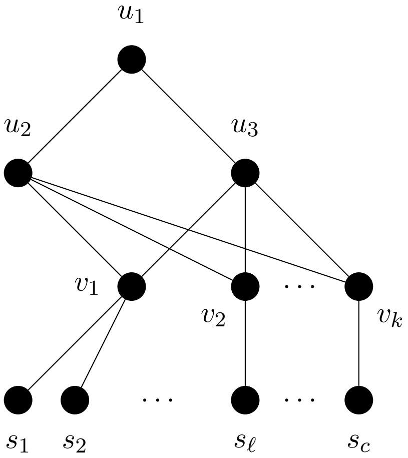

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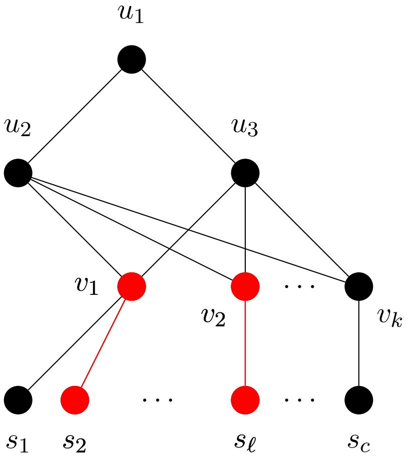

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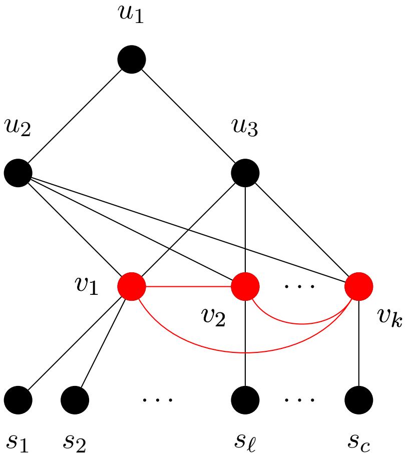

Theorem (Abuadas-C.-Hoàng-Sawada 2024): There are only finitely many k-critical $(P_3 + cP_1)$-free graphs for all $k \geq 1$ and $c \geq 0$.

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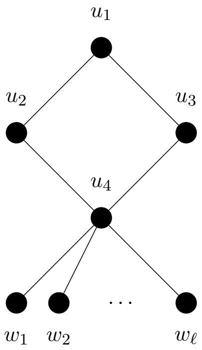

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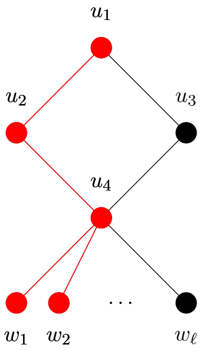

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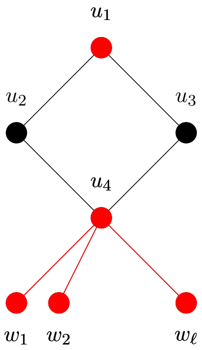
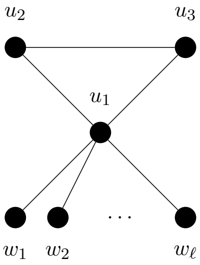


Figure: $(4, \ell)$-*squid* contains an induced chair and $claw + P_1$.

Theorem (Adekanye-Bury-C.-Knodel 2024): There are only finitely many k-critical $(2P_2$,$(3,\ell)$-$squid)$-free graphs for all $k, \ell \geq 1$.



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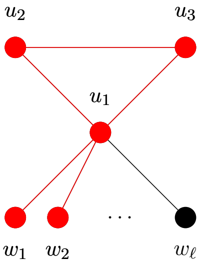

Corollary (Adekanye-Bury-C.-Knodel 2024): There are only finitely many k-critical *cricket*-free graphs for all $k \geq 1$.

Lemma (Adekanye-Bury-C.-Knodel 2024): Every k-critical $(2P_2, bull)$-free graph is $(P_3 + P_1)$-free.

Theorem (Adekanye-Bury-C.-Knodel 2024): There are only finitely many k-critical $(2P_2, bull)$-free graphs for all k.

Figure: The *bull* graph.

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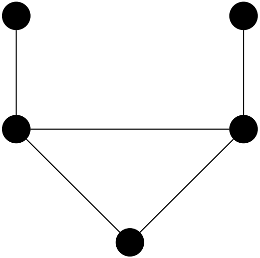

Figure: The *bull* graph.

n	4-critical	5-critical	6-critical	7-critical
4	1	0	0	0
5	0	1	0	0
6	1	0	1	0
7	2	1	0	1
8	0	2	1	0
9	0	11	2	1
10	0	0	12	2
11	0	0	126	12
12	0	0	0	128
13	0	0	0	3806
total	4	15	142	3947

Table: Number of k-critical $(2P_2, H)$-free graphs of order n for $k \leq 7$ where H is $(4,1)$-*squid* or *bull*.

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Question For which graphs H are there only finitely many k-critical (P_5, H)-free graphs for all k?

When H is order 5, only unknown for following graphs:

- co-gem
- chair (known $k = 5$)
- *cricket*
- $C_4 + P_1$
- bull (known $k = 5$)
- $\overline{P_3 + 2P_1}$
- W_4
- $K_5 - e$ (known $k \geq 8$)
- K_5 (known $k = 5$)

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Theorem (Beaton-C. 2024+): There are only finitely many k-critical $(P_5, P_3 + P_2, \text{co-gem})$-free graphs for all k.

THANK YOU!

ALBERTA INNOVATES
NSERC CRSNG