On the finiteness of k-vertex-critical $2P_2$-free graphs with forbidden induced squids or bulls

Ben Cameron (he/him)

University of Prince Edward Island

`brcameron@upei.ca`

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Yukon
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Quebec
Quebec
Charlottetown
Fredericton
Halifax
Prince Edward Island
Nova Scotia
New Brunswick
CANADA – Political
International border
Provincial/territorial border
Alberta Province/territory
Ottawa National capital
Regina Provincial/territorial capital
Scale
0 250 500 750 1000 km
0 250 500 750 miles


CHEF
CANADA No. 1
50 lb 22.7 kg
PRINCE EDWARD ISLAND
P.E.I.
POTATOES
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POMMES DE TERRE
CURRAN FARMS INC.
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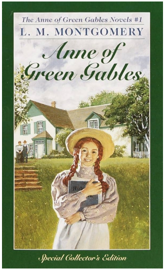
The Anne of Green Gables Novels #1
L. M. MONTGOMERY
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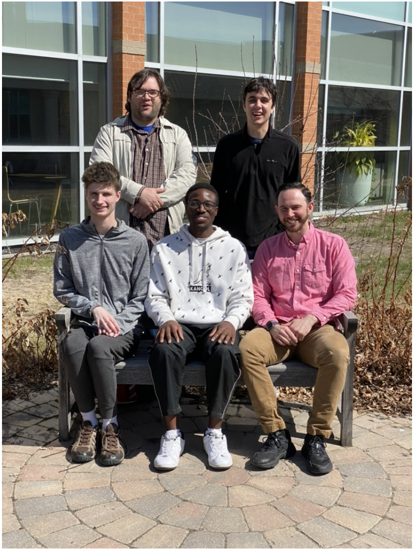
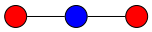
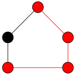


Figure: 2023 Summer Research Lab, including coauthors. (Joint work with Melvin Adekanye, Christopher Bury, and Thaler Knodel)

Definitions

- P_n is the path on n vertices (P_3: ).
- $G + H$ denotes the disjoint union of graphs G and H.
- $\ell G = \underbrace{G + G + \cdots + G}_{\ell}$ ($P_2 + 2P_1$: ).
- Colouring here means proper colouring (adjacent vertices get different colours).
- A graph is H-free if it does not contain H as an **induced subgraph**.  is P_5-free but not P_4-free.

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- So assuming P≠NP, if k-Colouring can be solved in polynomial-time for H-free graphs, then every component of H must be a path.
- It remains NP-complete when restricted to P_6-free graphs for all $k \geq 5$ (Huang 2016).

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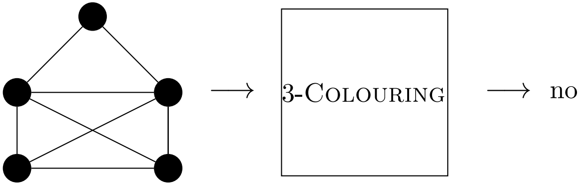

- A k-colouring is a certificate to verify a "yes".
- How can we verify a "no"?

- A graph G is k-critical if G is not $(k-1)$-colourable, but every proper induced subgraph of G is.

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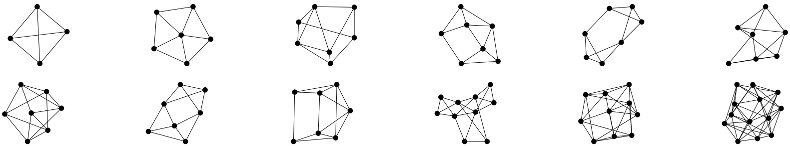
Issue: For $k \geq 3$ there are infinitely many k-critical graphs.

Theorem (Bruce-Hoàng-Sawada 2009, Maffray-Morel 2012): All twelve 4-critical P_5-free graphs are given below.



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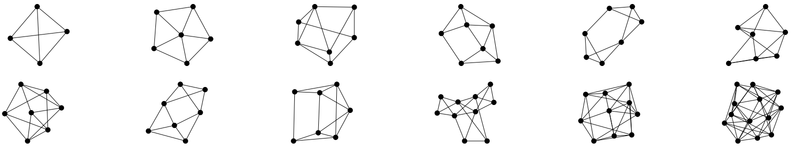

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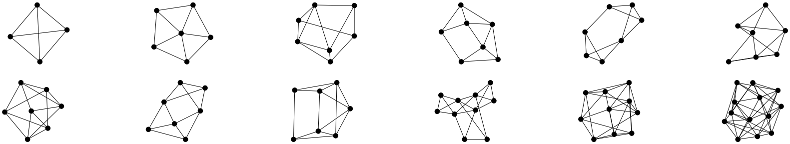

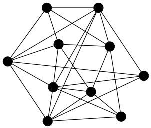

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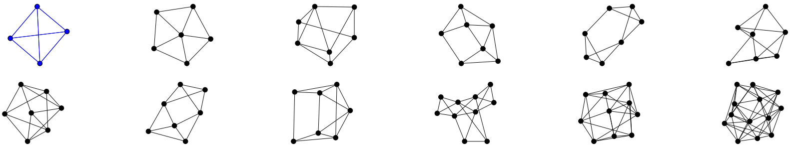

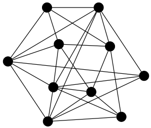

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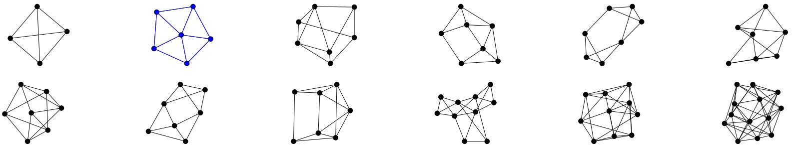

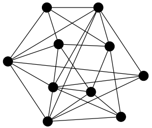

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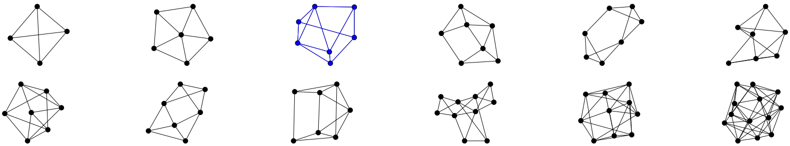

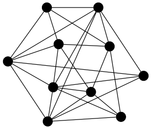

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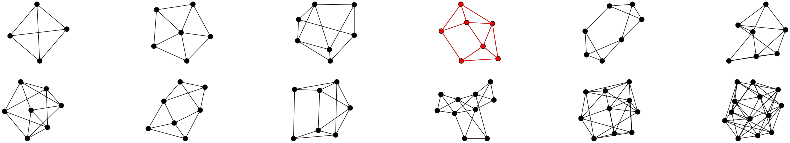

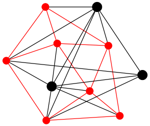

Question: Are there only finitely many k-critical P_5-free graphs for all k?

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Theorem (Hoàng-Moore-Recoskie-Sawada-Vatshelle 2015): There are infinitely many k-critical P_5-free graphs for all $k \geq 5$.

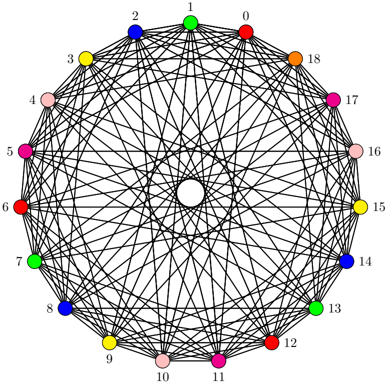

Figure: One of infinitely many 7-critical P_5-free graphs.

Question 1: For which graphs H are there only finitely many k-critical (P_5, H)-free graphs **for all** k?

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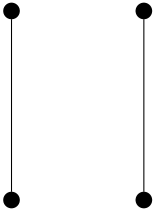
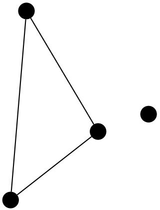
Theorem (K. Cameron-Goedgebeur-Huang-Shi 2021): For H of order 4 and $k \geq 5$, there are infinitely many k-critical (P_5, H)-free graphs if and only if H is $2P_2$ or $K_3 + P_1$.



(a) $2P_2$



(b) $K_3 + P_1$

Open Problem (K. Cameron-Goedgebeur-Huang-Shi 2021): For which graphs H of **order 5** are there only finitely many k-critical (P_5, H)-free graphs for all k?

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Finite if H is any of the graphs below:

- banner
- $\overline{P_5}$
- $K_{2,3}$ or $K_{1,4}$
- $\overline{P_3 + P_2}$ or gem
- $P_2 + 3P_1$
- $P_3 + 2P_1$
- $K_{1,3} + P_1$ or $\overline{K_3 + 2P_1}$

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- $P_2 + 3P_1$ (C.-Hoàng-Sawada 2022)
- $P_3 + 2P_1$ (Abuadas-C.-Hoàng-Sawada 2024)
- $\overline{P_5}$ (Dhaliwal et al. 2017)
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Theorem (C.-Hoàng 2024): There are infinitely many k-critical (P_5, C_5)-free graphs for all $k \geq 6$.

Recall...

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They prove this by constructing an infinite family of k-critical $(2P_2, K_3 + P_1)$-free graphs!

Figure: P_5 with an induced $2P_2$ in red.

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Theorem (C.-Hoàng 2024): There are infinitely many k-critical $(2P_2,\ K_3 + P_1,\ C_5)$-free graphs for all $k \geq 6$.

Fact: Every known infinite family of k-critical P_5-free graphs is actually $2P_2$-free!

Question: Does the finiteness of k-critical (P_5, H)-free graphs always coincide with that of k-critical $(2P_2, H)$-free graphs?

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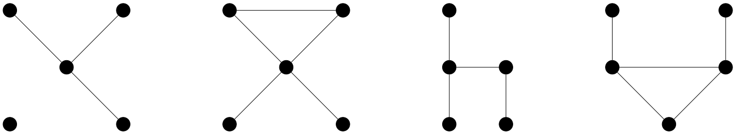


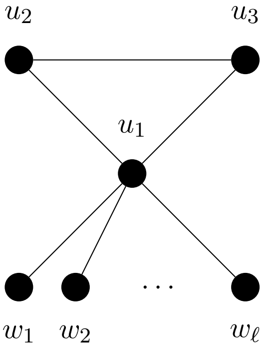
(a) $claw + P_1$ (b) $\overline{diamond + P_1}$ (c) $\overline{chair}$ (d) $bull$

Figure: Graphs H of order 5 where the finiteness of k-critical (P_5, H)-free graphs is unknown.

(a) $m = 3$.

(b) $m = 4$.

Figure: The general form of the (m, ℓ)-*squid* graphs for $m = 3, 4$.



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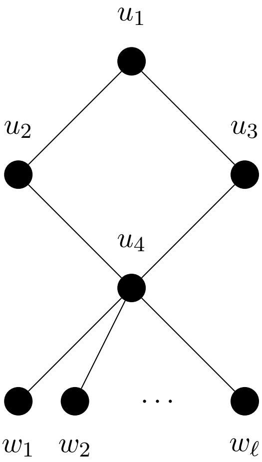

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Figure: The general form of the (m, ℓ)-*squid* graphs for $m = 3, 4$.

Lemma (IWOCA2024): Let $\ell, k \geq 1$ and $c = (\ell - 1)(k - 1) + 1$. If G is a k-critical $(2P_2$, $(4, \ell)$-*squid*$)$-free graph, then G is $(P_3 + cP_1)$-free.

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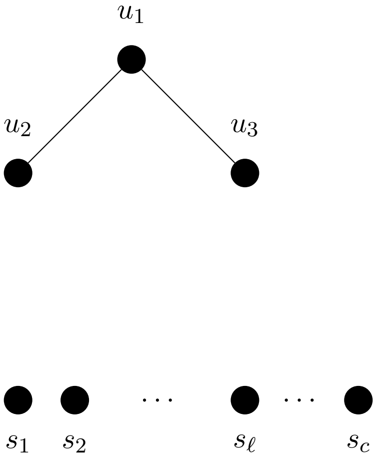

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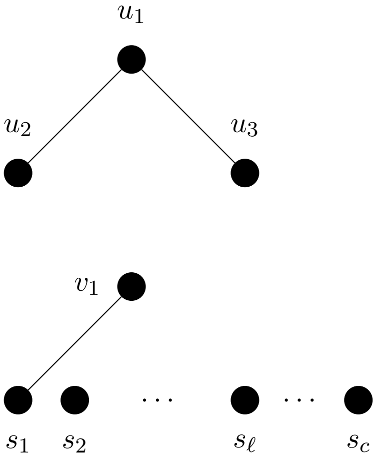

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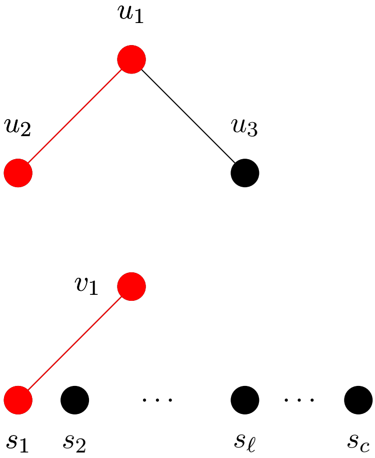

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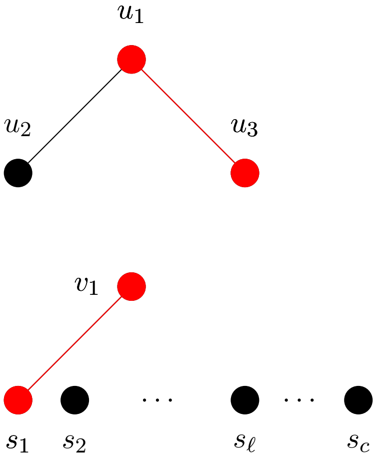

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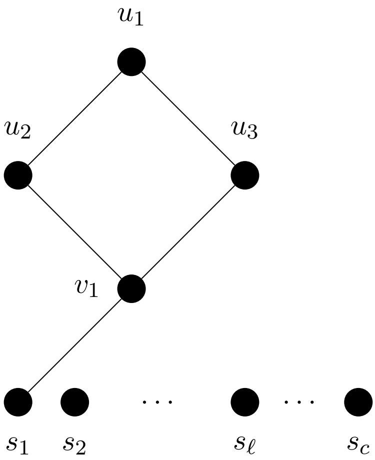

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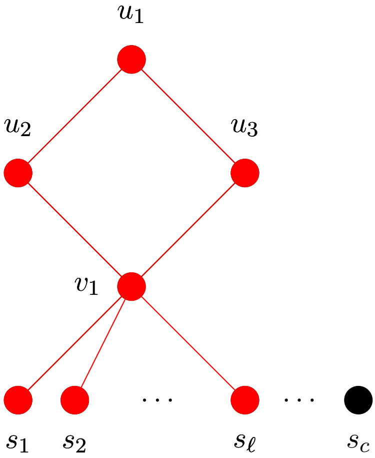

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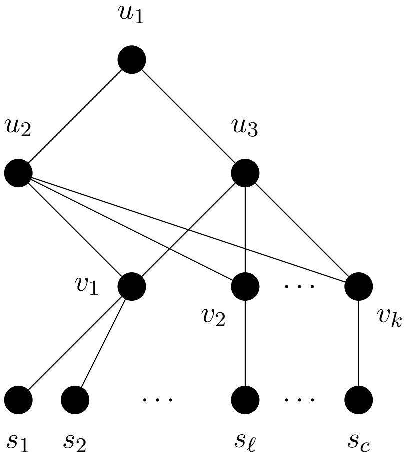

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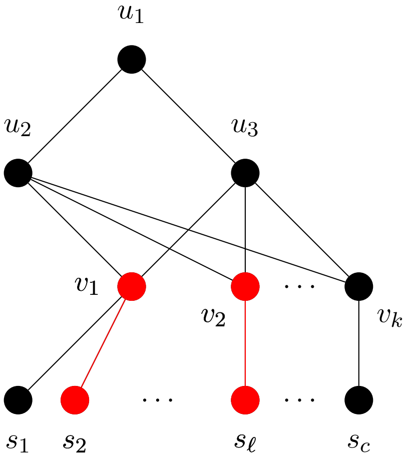

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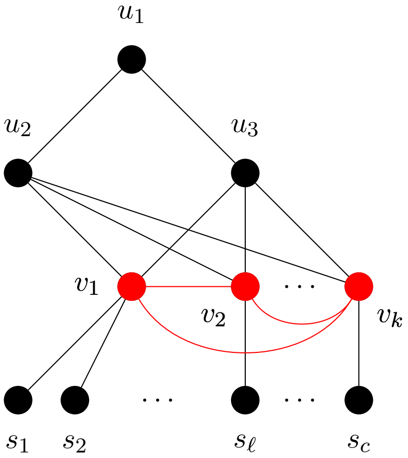

Theorem (Abuadas-C.-Hoàng-Sawada 2024): There are only finitely many k-critical $(P_3 + cP_1)$-free graphs for all $k \geq 1$ and $c \geq 0$.

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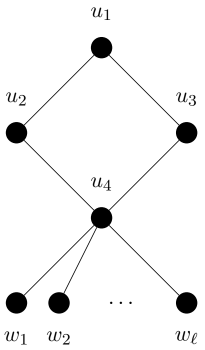

Figure: $(4,\ell)$-*squid* contains an induced

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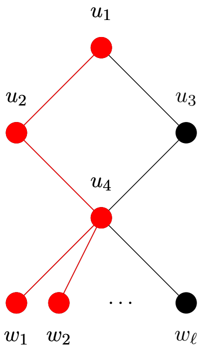

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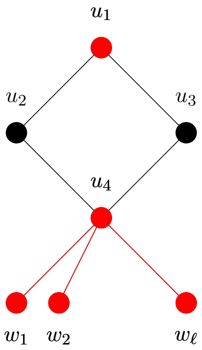
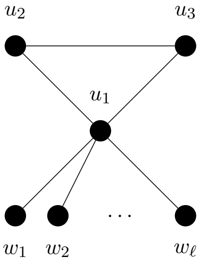


Figure: $(4, \ell)$-*squid* contains an induced chair and $claw + P_1$.

Lemma (IWOCA2024): Let $\ell, k \geq 1$. If G is k-critical $(2P_2$, $(3,\ell)$-*squid*$)$-free, then G is $(4, 2\ell-1)$-*squid*-free.



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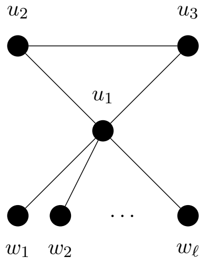

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Corollary (IWOCA2024): There are only finitely many k-critical ($\overline{diamond + P_1}$)-free graphs for all $k \geq 1$.

Lemma (IWOCA2024): Every k-critical $(2P_2, bull)$-free graph is $(P_3 + P_1)$-free.

Theorem (IWOCA2024): There are only finitely many k-critical $(2P_2, bull)$-free graphs for all k.

Figure: The *bull* graph.

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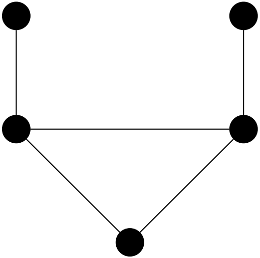

Figure: The *bull* graph.

n	4-critical	5-critical	6-critical	7-critical
4	1	0	0	0
5	0	1	0	0
6	1	0	1	0
7	2	1	0	1
8	0	2	1	0
9	0	11	2	1
10	0	0	12	2
11	0	0	126	12
12	0	0	0	128
13	0	0	0	3806
total	4	15	142	3947

Table: Number of k-critical $(2P_2, H)$-free graphs of order n for $k \leq 7$ where H is $(4,1)$-*squid* or *bull*.

Question: Does the finiteness of k-critical (P_5, H)-free graphs always coincide with that of k-critical $(2P_2, H)$-free graphs?

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Question For which graphs H are there only finitely many k-critical (P_5, H)-free graphs for all k?

When H is order 5, only unknown for following graphs:

- $P_4 + P_1$
- chair (known $k = 5$)
- $\overline{\text{diamond} + P_1}$
- $C_4 + P_1$
- bull (known $k = 5$)
- $\overline{P_3 + 2P_1}$
- W_4
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Thank You!

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