Critical (P_5, W_4)-free graphs

Ben Cameron (he/him)

University of Prince Edward Island

`brcameron@upei.ca`

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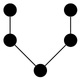
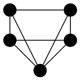
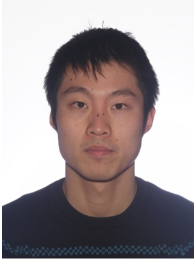


May 21, 2025



Joint work with Wen Xia, Shenwei Huang, Jan Goedgebeur, Jorik Jooken, & Iain Beaton

(a) Wen Xia



(b) Shenwei Huang

(c) Jorik Jooken

(d) Jan Goedgebeur

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- How can we verify a “no”?

- A graph G is *k-critical* if G is not $(k-1)$-colourable, but every proper induced subgraph of G is.

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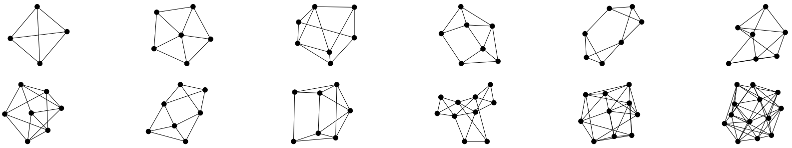
$\Longrightarrow$ Such an algorithm exists when there are only finitely many k-critical graphs in the hereditary family.

Theorem (Bruce-Hoàng-Sawada 2009, Maffray-Morel 2012): All twelve 4-critical P_5-free graphs are given below.



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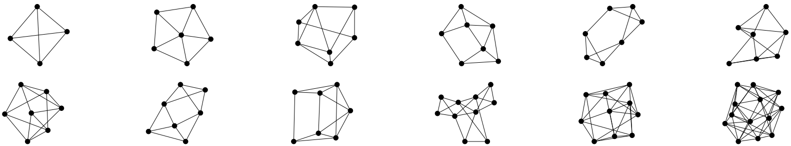

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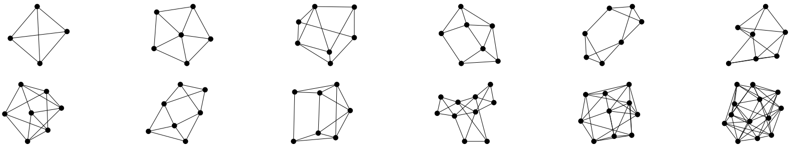

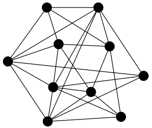

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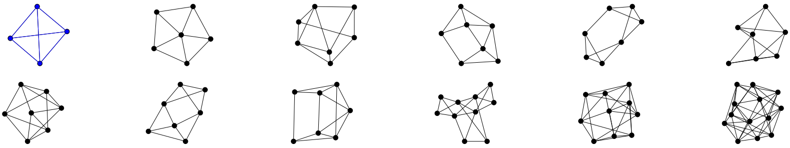

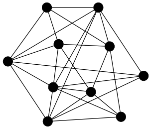

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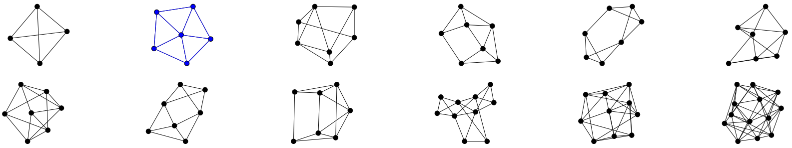

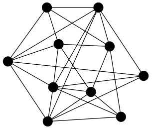

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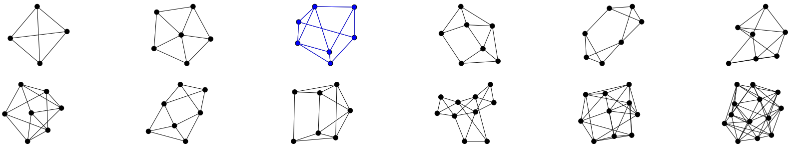

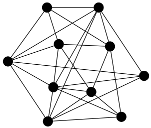

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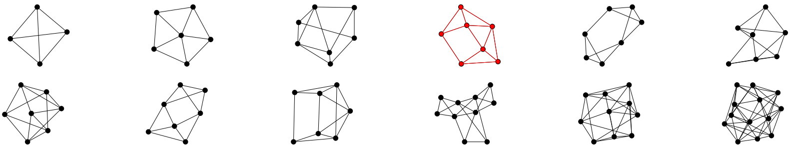

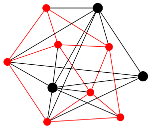

Reasonable Question: Are there only finitely many k-critical P_5-free graphs for all k?

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Question 1: For which graphs H are there only finitely many k-critical (P_5, H)-free graphs for all k?

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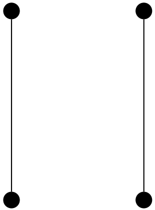
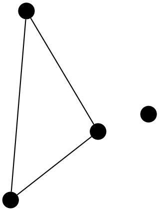
Theorem (K. Cameron-Goedgebeur-Huang-Shi 2021): For H of order 4 and $k \geq 5$, there are only finitely many k-critical (P_5, H)-free graphs if and only if H is **NOT** $2P_2$ or $K_3 + P_1$.



(a) $2P_2$



(b) $K_3 + P_1$

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True if H is any of the graphs below:

- $\overline{K_5}$
- banner
- $K_{2,3}$ or $K_{1,4}$
- $P_2 + 3P_1$
- $P_3 + 2P_1$
- $\overline{P_5}$
- $\overline{P_3 + P_2}$ or gem
- dart
- $K_{1,3} + P_1$ or $\overline{K_3 + 2P_1}$

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- $\overline{K_5}$ (Ramsey 1928)
- banner (Brause-Geißer-Schiermeyer 2022)
- $K_{2,3}$ or $K_{1,4}$ (Kamiński-Pstrucha 2019)
- $P_2 + 3P_1$ (C.-Hoàng-Sawada 2022)
- $P_3 + 2P_1$ (Abuadas-C.-Hoàng-Sawada 2024)
- $\overline{P_5}$ (Dhaliwal-Hamel-Hoàng-Maffray-McConnel-Panait 2017)
- $\overline{P_3 + P_2}$ or gem (Cai-Goedgebeur-Huang 2023)
- dart (Xia-Jooken-Goedgebeur-Huang 2023)
- $K_{1,3} + P_1$ or $\overline{K_3 + 2P_1}$ (Xia-Jooken-Goedgebeur-Huang 2024)

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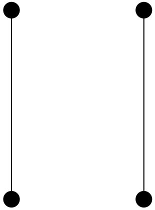
Theorem (K. Cameron-Goedgebeur-Huang-Shi 2021): There are infinitely many k-critical (P_5, H)-free graphs for all $k \geq 5$ if $H = 2P_2$ or $K_3 + P_1$.



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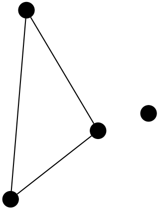

(b) $K_3 + P_1$

Theorem (C.-Hoàng 2024): There are infinitely many k-critical (P_5, C_5)-free graphs for all $k \geq 6$.

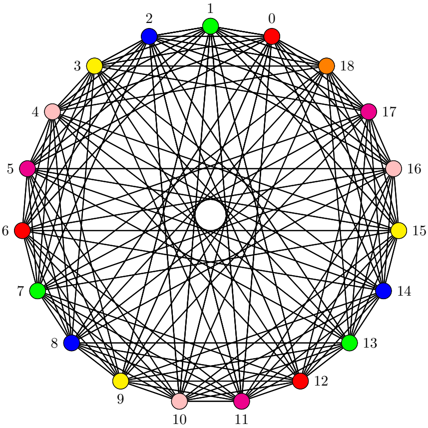

Figure: A 7-critical graph (P_5, C_5)-free graph from the infinite family.

Thus, the Open Problem is only open for the following graphs:

- co-gem
- chair (known $k = 5$)
- cricket
- $C_4 + P_1$
- bull (known $k = 5$)
- $\overline{P_3 + 2P_1}$
- W_4
- $K_5 - e$ (known $k \geq 8$)
- K_5 (known $k = 5$)

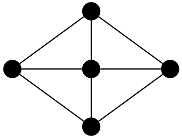

Figure: The 4-wheel W_4.

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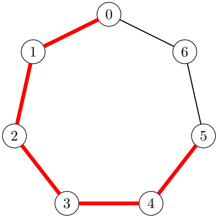
- Every P_5-free graph is also C_{2k+1}-free for all $k \geq 3$.



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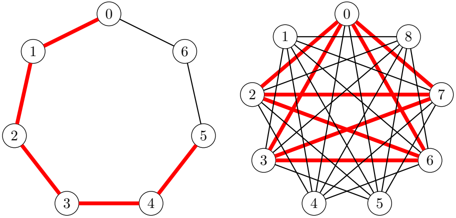

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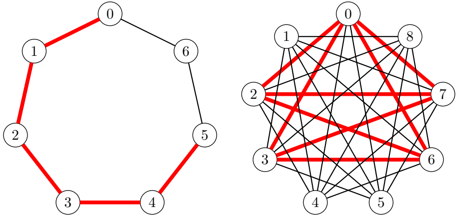
- Every W_4-free graph is also $\overline{C_{2k+1}}$-free for all $k \geq 3$.
- Thus, a k-critical (P_5, W_4)-free graph must be K_k or contain an induced C_5 or $\overline{C_7}$.



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Proof Ideas (cont.): Build from an induced substructure C for $C = C_5$ and $C = \overline{C_7}$ and partition the rest of the vertices by their neighbours on C. Then determine which of these sets are:

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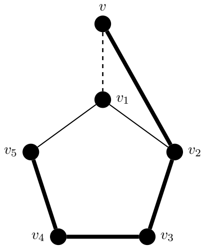
- Empty



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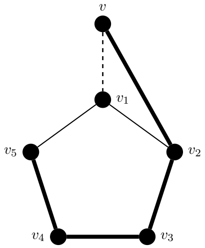

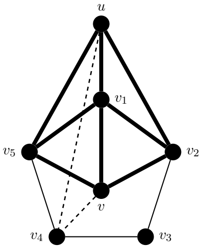

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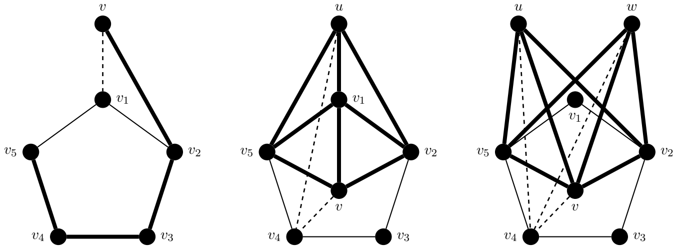
- Cliques
- P_3-free



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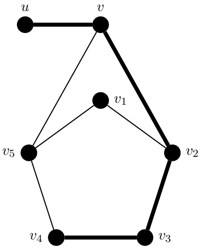

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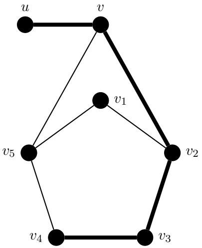
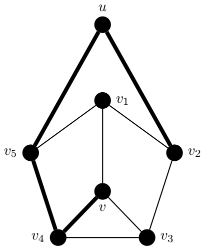
- Complete

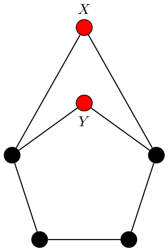




Proof Ideas (cont.):

Lemma (K. Cameron-Goedgebeur-Huang-Shi 2021): Let G be a k-critical graph. Then G has no two nonempty disjoint subsets X and Y of $V(G)$ that satisfy all the following conditions.

- X and Y are anticomplete to each other.
- $\chi(G[X]) \leq \chi(G[Y])$.
- Y is complete to $N(X)$.



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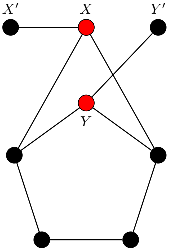

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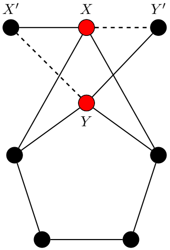


Using an exhaustive generation algorithm we obtain:

n	# 5-critical	# 5-edge-critical
5	1	1
6	0	0
7	1	1
8	1	1
9	44	7
10	4	1
11	0	0
12	1	1
13	8	6
14	0	0
15	2	1
16	0	0
17	2	2
Total	64	21

Table: An overview of the number of pairwise non-isomorphic 5-critical (P_5, W_4)-free graphs and 5-edge-critical (P_5, W_4)-free graphs.

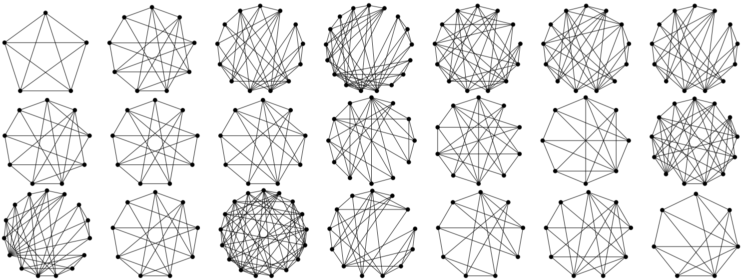
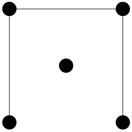


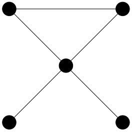
Figure: All 5-edge-critical (P_5, W_4)-free graphs.

Open Problem (K. Cameron-Goedgebeur-Huang-Shi 2021): For which graphs H of **order 5** are there only finitely many k-critical (P_5, H)-free graphs for all k?

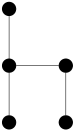
Now only open for following graphs:



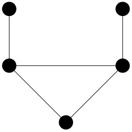
(a) co-gem



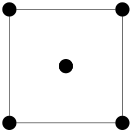
(b) cricket



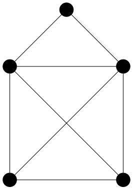
(c) chair



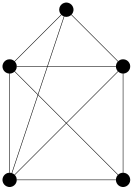
(d) bull



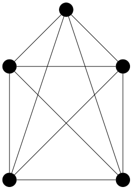
(e) $C_4 + P_1$



(f) $\overline{P_3 + 2P_1}$



(g) $K_5 - e$



(h) K_5

Recall: Since W_4 is an induced subgraph of $\overline{C_{2\ell+1}}$ for all $\ell \geq 4$, every imperfect k-critical (P_5, W_4)-free graph must contain an induced C_5 or $\overline{C_7}$.

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The following are graphs of order 5 are induced subgraphs of $\overline{C_{2\ell+1}}$ for all:

$\ell \geq 3$	$\ell \geq 4$	$\ell \geq 5$
$\overline{P_5}$	W_4	K_5
$\overline{P_4 + P_1}$	$\overline{P_3 + 2P_1}$	
$\overline{P_3 + P_2}$	$K_5 - e$	

Table: There are only finitely many k-critical (P_5, H)-free graphs for all of the blue graphs H above.

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Table: There are only finitely many k-critical (P_5, H)-free graphs for all of the blue graphs H above.

- The other 5 cases *may* require different proof techniques...

Theorem (Abuadas-C.-Hoàng-Sawada 2024): There are only finitely many k-critical $(P_3 + cP_1)$-free graphs for all $k \geq 1$ and $c \geq 0$.

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Theorem (Adekanye-Bury-C.-Knodel 2024): There are only finitely many k-critical $(2P_2,\ (4,\ell)\text{-}squid)$-free graphs for all $k, \ell \geq 1$.

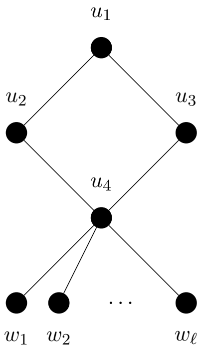

Figure: $(4, \ell)$-*squid*

Theorem (Abuadas-C.-Hoàng-Sawada 2024): There are only finitely many k-critical $(P_3 + cP_1)$-free graphs for all $k \geq 1$ and $c \geq 0$.

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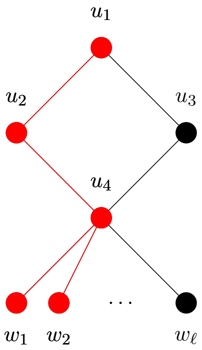
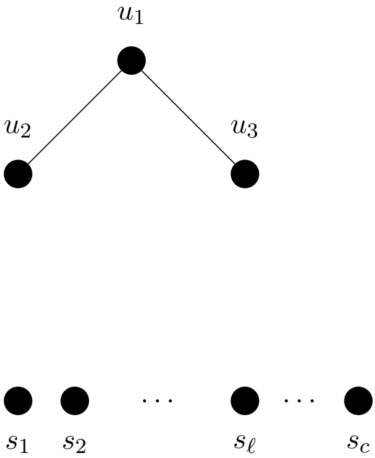


Figure: $(4,\ell)$-*squid* contains an induced chair.

- Proved by showing every k-critical ($2P_2$, $(4,\ell)$-*squid*)-free graph is $(P_3 + cP_1)$-free (for $c = (\ell - 1)(k - 1) + 1$).

- Proved by showing every k-critical ($2P_2$, $(4, \ell)$-*squid*)-free graph is $(P_3 + cP_1)$-free (for $c = (\ell - 1)(k - 1) + 1$).



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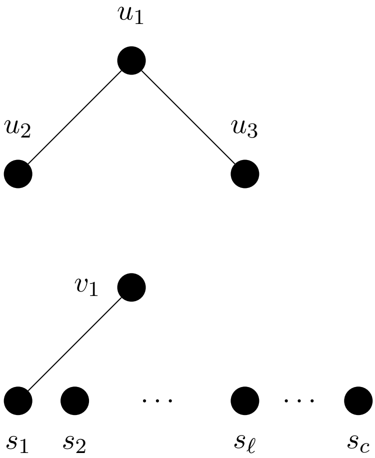

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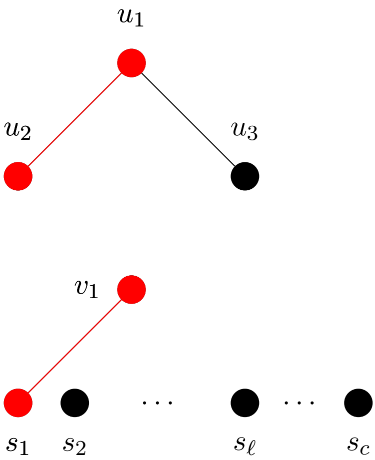

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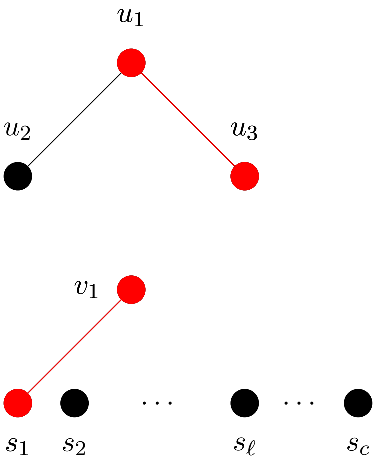

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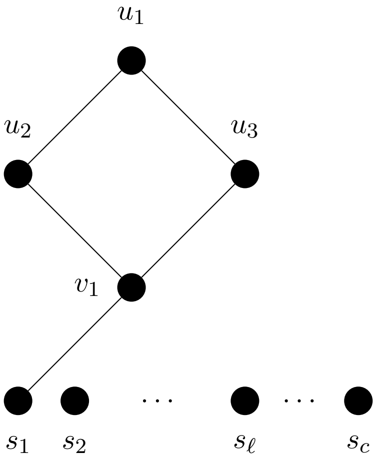

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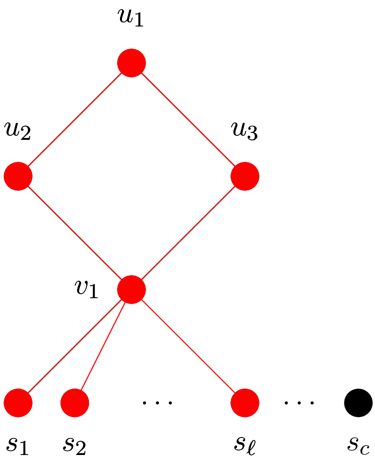

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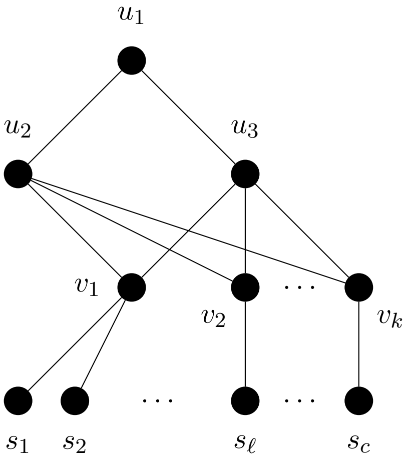

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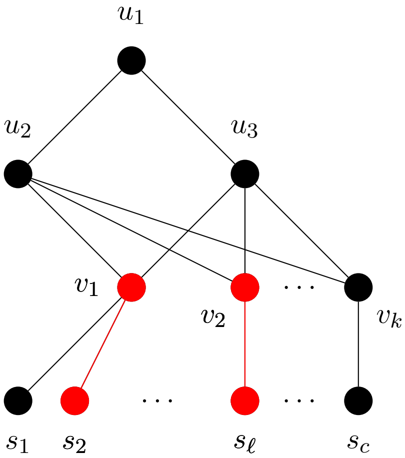

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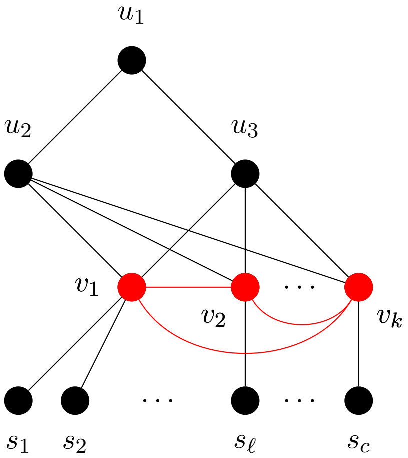

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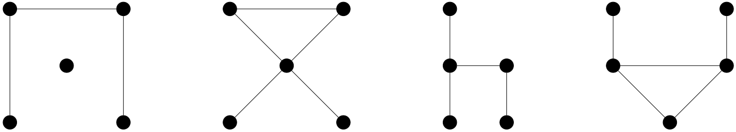
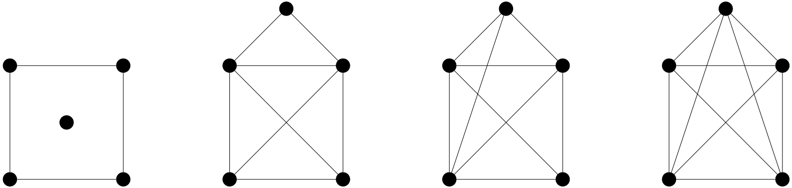


(a) co-gem (b) cricket (c) chair (d) bull



(e) $C_4 + P_1$ (f) $\overline{P_3 + 2P_1}$ (g) $K_5 - e$ (h) K_5

Thank You!

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